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On the relationship of gravitational constants in KK reduction

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Abstract: In this short note, we try to clarify a seemly trivial but often confusing question in relating a higher-dimensional *physical* gravitational constant to its lower-dimensional correspondence in Kaluza-Klein reduction. In particular, we re-derive the low-energy M-theory gravitational constant in terms of type-IIA string coupling g_s and constant α' through the metric relation between the two theories.

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The proper determination of eleven dimensional M-theory gravitational constant (therefore, the eleven-dimensional Planck constant), in terms of type-IIA string coupling g_s and constant α', is important, for example, for the BFSS matrix proposal of M-theory [1]. It is also important for whether brane modes can possibly decouple from bulk gravity modes [2, 3, 4] in the so-called decoupling limit. Given the string constant α' (therefore the units in type-IIA string theory) and the relationship between 11-D M-theory and type-IIA string theory, the 11-D M-theory *physical* gravitational constant as well as the units for M-theory are also given. We therefore expect a precise expression for the M-theory gravitational constant in terms of type-IIA string coupling g_s and constant α'. However, there exist no unique answers in the literature for this constant. We try to clarify, in this short note, possible confusion about the derivations of this constant.

Let us begin with a general discussion in relating a higher-dimensional physical gravitational constant to its lower-dimensional correspondence in dimension redection. We start with the following gravity action in dimension D

$$I_D = \frac{1}{2\bar{\kappa}_D^2} \int d^D x \sqrt{-\det \hat{G}} \left[\hat{R} + \cdots\right], \tag{1}$$

where $\hat{G}_{MN}$ is the metric, $\hat{R}$ is the Ricci scalar, the constant $\bar{\kappa}_D^2$ is usually called gravitational constant (the Newton constant $8\pi G_N^D \equiv \kappa_D^2$),[1] and $\cdots$ in the above action represents other possible fields.[2]

Now we wish to compactify the above action to dimension d ($< D$). For our purpose, we need to consider only the massless graviton whose effective action is

$$I_d = \frac{1}{2\bar{\kappa}_d^2} \int d^d x \sqrt{-\det g} \left[R + \cdots\right]. \tag{2}$$

In obtaining the above action from eq. (1), we made the split for the higher-dimensional coordinate $x^M = (x^\mu, y^i)$ with $M = 0, 1, \ldots, D-1$, $\mu = 0, 1, \ldots, d-1$ and $i = 1, \ldots, D-d$. Here x^μ are the coordinates of the lower-dimensional spacetime and y^i are the compactified coordinates. We therefore have the same units for both the D-dimensional theory and the compactified d-dimensional theory. The massless sector of the lower-dimensional theory can be obtained by assuming the higher-dimensional fields to be independent of y^i. We then simply integrate out the y^i from the action (1). By comparing the resulted action with eq. (2), we have the relation for the gravitational constants and the compactified volume V_{D-d} as[3]

$$\kappa_D^2 = \kappa_d^2 V_{D-d}\,. \tag{3}$$

[1]The κ without a bar corresponds to the physical gravitional constant while the one with a bar is not necesarily a physical one, see the explanation given in the text.

[2]For the purpose of this paper, we need to consider only the first term in the action.

[3]I will make this precise when we discuss on how to reduce the low energy M-theory to the low energy type-IIA string theory.

We often say that the above equation, relating the higher-dimensional physical gravitational constant to its lower-dimensional correspondence through the physical volume measured with respect to the lower-dimensional metric, is independent of the actual metric relation between the higher-dimensional and the lower-dimensional theories. This is true indeed.

However, what is often ignored in practice is the implicit assumption used in deriving eq. (3) that we choose the asymptotic metric (i.e., the underlying vacuum) for the higher-dimensional theory to be same as that for the compactified lower-dimensional theory.[4] Only in this case, we can take the constant $\bar{\kappa}^2$ in front of the respective action as the physical gravitational constant.[5] In general, however, the asymptotic metrics for both the higher-dimensional theory and the compactified lower-dimensional theory are not necessarily the same because the scalars due to the compactification develop VEV.[6] If this happens, we cannot take both $\bar{\kappa}_D^2$ and $\bar{\kappa}_d^2$ in front of the respective action in the above as physical.[7] The ignorance of this fact is often the source of confusion in the literature. For example, the low-energy M-theory physical gravitational constant has been given correctly in [1, 7] as $2\kappa_{11}^2 = (2\pi)^8 g_s^3 \alpha'^{9/2}$ in terms of type-IIA string coupling g_s and constant α'. But this constant has also been given incorrectly in the literature precisely because $\bar{\kappa}$ is mistaken as κ.

In the remainder of this note, I will focus, as an example, on the reduction of 11-dimensional low-energy M-theory on a circle S^1 to give the low energy theory of type-IIA string. The 11-D M-theory metric is related to the type-IIA string metric as

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3}(dx^{11})^2\,, \tag{4}$$

where ϕ is the dilaton in type-IIA string theory, 11-th coordinate x^{11} has a period $2\pi r$ with the coordinate radius r. In the above we have dropped the KK vector

[4] Usually we choose the asymptotic metric for the lower-dimensional theory to be minkowskian, i.e., flat Minkowski metric $\eta_{\mu\nu} = (-, +, \ldots, +)$, see the definition given in [7]. This is the metric used in defining the *physical* gravitational constant. This is also the metric used in perturbative string theory in defining the string tension $T_f = 1/(2\pi\alpha')$ or the string constant α'.

[5] In this case, for example, the metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ for small graviton fluctuation around flat Minkowski spacetime and the Einstein-Hilbert action reduces to the cannonical form $\int (\partial h)^2 + \kappa(\partial h)^2 h$.

[6] Of course, one can always define both of the higher-dimensional metric and the lower-dimensional one to be same asymptotically by absorbing the possible constant factor due to the VEV of scalars into the $\bar{\kappa}^2$ in front of the action. For the higher-dimensional theory, the compactified coordinates y^i have to be rescaled properly with respect to the lower-dimensional metric, see the example given later in relating M-theory to type-IIA string. Then the resulting κ^2 is the physical one. This is what Maldacena did in [5] for obtaining masses properly for BPS states in string theory using U-duality.

[7] Polchinski in [6] chose both the units and metric for M-theory the same as those for string theory. By definition, his κ_{11}^2 and the compactified radius are both physical, not the usual $\bar{\kappa}_{11}^2$ and the coordiante radius r.

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field A_μ. For our purpose, A_μ is irrelevant. We take type-IIA string metric ds^2_{10} as asympotically Minkowski since that is where we quantize the perturbative type-IIA string. That is the metric used in defining the fundamental string tension $T_f = 1/(2\pi\alpha')$. So are the tensions for D-branes and NSNS branes. With eq. (4) and taking $D = 11$ in eq. (1), we have the low energy action of type-IIA string as

$$I_{10} = \frac{2\pi r}{\bar{\kappa}^2_{11}} \int d^{10}x \sqrt{-\det g} e^{-2\phi} \left[R + \cdots\right], \tag{5}$$

where $g_{\mu\nu}$ is the string metric. By definition, we have the following relation

$$\bar{\kappa}^2_{11} = 2\pi r\, e^{-2\phi_0}\, \kappa^2_{10}\,, \tag{6}$$

where $\kappa^2_{10} \equiv \bar{\kappa}^2_{10}\, e^{2\phi_0}$ is the physical gravitational constant in D = 10. ϕ_0 is the VEV of dilaton or the asymptotic value of the dilaton and is related to the string coupling as $g_s = e^{\phi_0}$.

As we stress above that eq. (3) holds true always. In the present context, it is

$$\kappa^2_{11} = 2\pi\rho\, \kappa^2_{10}\,, \tag{7}$$

with ρ the physical radius. Let me explain why $2\kappa^2_{11} = (2\pi)^8 g_s^3 \alpha'^{9/2}$ given in [1, 7] must be correct. As I mentioned above, eq. (7) should hold always true. The physical gravitational constant $2\kappa^2_{10} = g_s^2\alpha'^4$ was given in [8]. As we now know that the strong coupling of type-IIA string is just M-theory compactified on a big circle. In order for this to be true, one needs to identify the spectrum of D0 branes with that of momentum (Kaluza-Klein) states. This implies that the physical radius of the circle measured in string metric is given as the inverse of mass of a single D0 brane, i.e., $\rho = g_s\alpha'^{1/2}$. Then we have the 11-D physical gravitational constant from eq. (7) as given above.

From eqs. (6) and (7), we have

$$\frac{\bar{\kappa}^2_{11}}{r\, e^{-2\phi_0}} = \frac{\kappa^2_{11}}{\rho}\,. \tag{8}$$

We intend to determine the relation between κ_{11} and $\bar{\kappa}_{11}$ and the relation between r and ρ unambiguously. To my knowledge, no explicit derivations for these two relations have been given in the literature. We dare to present them here.

For our purpose, we need to consider only the asymptotic metric relation in eq. (4), i.e.,

$$\begin{aligned}(ds_0)^2_{11} &= e^{-2\phi_0/3}(ds_0)^2_{10} + e^{4\phi_0/3}(dx^{11})^2\,,\\ &= e^{-2\phi_0/3}\left[(ds_0)^2_{10} + (d\tilde{x}^{11})^2\right],\end{aligned} \tag{9}$$

where the asymptotic string metric $(ds_0)^2_{10}$ is actually minkowskian and the rescaled 11-th coordinate $\tilde{x}^{11} = e^{\phi_0}x^{11}$ with its radius $\tilde{r} = e^{\phi_0}r$. The first line in the above equation indicates clearly that the 11-D metric cannot be asymptotically minkowskian if we insist ds_{10}^2 be so.[8] The second line says that the 11-D metric can be made asymptotically minkowskian up to a constant scaling factor $e^{-2\phi_0/3}$ if we rescale x^{11} to $\tilde{x}^{11}$ as given above. In other words, the scaled radius $\tilde{r}$ is measured with respect to the string metric. Because the string constant α' is defined with respect to the string metric, the second line in eq. (9) should be used in the following equation.

By definition, from the above asymptotic metric relation, we have

$$\kappa_{11}^2 = \frac{\bar{\kappa}_{11}^2}{\sqrt{-\det \hat{G}_0 \hat{G}_0^{-1}}} = e^{3\phi_0}\, \bar{\kappa}_{11}^2\,, \tag{10}$$

where $\hat{G}_0$ denotes the 11-D asymptotic metric given in the second line of eq. (9). Using the above and eq. (8), we derive $r = \alpha'^{1/2}$. Then $\tilde{r} = e^{\phi_0}\, \alpha'^{1/2} = \rho$ is the physical radius measured in the string metric.

Let us provide an independent check of eq. (10). For simplicity, we consider the reduction of a scalar field $\Phi(x^M)$ from 11-D to 10-D on a circle S^1. The usual KK reduction says

$$\Phi(x^\mu, x^{11}) = \sum_{n=-\infty}^{\infty} \Phi_n(x^\mu) e^{inx^{11}/r}\,, \tag{11}$$

or

$$\Phi(x^\mu, \tilde{x}^{11}) = \sum_{n=-\infty}^{\infty} \Phi_n(x^\mu) e^{in\tilde{x}^{11}/\tilde{r}}\,, \tag{12}$$

where x^{11} or $\tilde{x}^{11}$ is the compactified coordinate defined earlier. The 11-D wave equation $\bigtriangledown_M \bigtriangledown^M \Phi = 0$ in the asymptotic region (or around the Minkowski vacuum) becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi_n(x^\mu) = \frac{n^2}{r^2 e^{2\phi_0}}\Phi_n(x^\mu)\,, \tag{13}$$

where we have used the first line in eq. (9), or we have

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi_n(x^\mu) = \frac{n^2}{\tilde{r}^2}\Phi_n(x^\mu)\,, \tag{14}$$

where we have used the second line in eq. (9). The mass spectrum with respect to the 10-D Minkowski vacuum in string frame can be obtained from either of the above equations as

$$M_n^2 = -p_\mu p^\mu = \frac{n^2}{r^2 e^{2\phi_0}} = \frac{n^2}{\tilde{r}^2}\,. \tag{15}$$

From the above, we should identify $\tilde{r} = re^{\phi_0}$ as the physical radius with respect to the string metric. The mass M_n should be identified with that of n D0 brane for the

[8] We can no longer rescale $(ds_0)_{10}^2$ since that is the metric used in defining the string constant α'.

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reason mentioned earlier. We therefore have $\tilde{r} = g_s \alpha'^{1/2}$. So we have $r = \alpha'^{1/2}$. Using eq. (8) and $\rho = \tilde{r}$, we obtain also eq. (10). Our discussion gives $2\bar{\kappa}_{11}^2 = (2\pi)^8 \alpha'^{9/2}$.

In summary, we explain how to obtain the physical gravitational constant for the original higher dimensional theory if we know the physical gravitational constant in the compactified lower-dimensional theory. In particular, we derive the relation $\kappa_{11}^2 = g_s^3 \bar{\kappa}_{11}^2$ and determine the radius $r = \alpha'^{1/2}$ (or $\tilde{r} = g_s \alpha'^{1/2}$) unambiguously.

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